Recurring Satistics

Paolo Bosetti

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Rationale

This memo reports the calculation scheme that can be adopted for calculating sample mean and variance statistics by using a pair of recurrence formulas. This approach comes handy whenever you have to perform a continuous, inline assessment of those indicators with minimum memory footprint (e.g. on microcontrollers), without the need of storing the whole set of sample values.

Sample Mean

This is an easy one: given the stochastic variable x and by indicating its sample mean for a set of i observations as $\bar{x}_i$, we have:

$$\begin{aligned} \bar{x}_1 &:= && x_1 && (1)\\ \bar{x}_n &= && \tfrac{1}{n}\sum_{i=1}^{n} x_i && (2)\\ &= && \tfrac{n-1}{n}\left(\sum_{i=1}^{n-1} \tfrac{x_i}{n-1}\right) + \tfrac{x_n}{n} && (3)\\ &= && \tfrac{1}{n}\left((n-1)\bar{x}_{n-1} + x_n\right) && (4) \end{aligned}$$

where x_n is the current observation (after n events), and x_1 is the very first observation.

By using the last equation for $\bar{x}_n$, the sample mean value $\bar{x}_n$ can be continuously updated at every acquisition using only the last observation x_n, the previous value of the sample mean $\bar{x}_{n-1}$, and the total number of observations n. There is **no need for storing the whole set of observations**, and the algorithm complexity is $O(1)$.

Sample variance

The recurrence formula for sample variance is a little more complex, and care must be payed in the formulation in order to avoid differences between small quantities, which may bring to large rounding errors.

By definition of sample variance for n observations, s_n^2:

$$s_n^2 = \sum_{i=1}^{n} \frac{(\bar{x}_n - x_i)^2}{n-1} = \frac{SS_n}{n-1} \tag{5}$$

$$= \frac{1}{n-1} \sum_{i=1}^{n} (x_i^2 - 2x_i\bar{x} + \bar{x}^2) \tag{6}$$

$$= \frac{1}{n-1} \left(\sum_{i=1}^{n} x_i^2 - \bar{x}^2 \right) \tag{7}$$

$$s_n = \sqrt{\frac{1}{n-1} \left(\sum_{i=1}^{n} x_i^2 - \frac{1}{n} \left(\sum_{i=1}^{n} x_i \right)^2 \right)} \tag{8}$$

The sum of squares SS_n (which is the only part in the definition of sample variance that is depending on previous values) can be thus be expressed as:

$$SS_n = \left(\sum_{i=1}^{n} x_i^2 - \frac{1}{n} \left(\sum_{i=1}^{n} x_i \right)^2 \right) \tag{9}$$

so that the increment in sum of squares can be obtained:

$$SS_n - SS_{n-1} = \sum_{1=1}^{n} x_i^2 - \frac{1}{n} \left(\sum_{1=1}^{n} x_i \right)^2 - \sum_{1=1}^{n-1} x_i^2 + \frac{1}{n-1} \left(\sum_{1=1}^{n-1} x_i \right)^2 \tag{10}$$

where we can substitute:

$$\sum_{1=1}^{n} x_i = n\bar{x}_n \tag{11}$$

$$\sum_{1=1}^{n-1} x_i = \sum_{1=1}^{n} x_i - x_n = n\bar{x}_n - x_n \tag{12}$$

$$\sum_{1=1}^{n} x_i^2 - \sum_{1=1}^{n-1} x_i^2 = x_n^2 \tag{13}$$

thus having:

$$SS_n - SS_{n-1} = x_n^2 - \frac{1}{n}(n\bar{x}_n)^2 + \frac{1}{n-1}(n\bar{x}_n - x_n)^2 \tag{14}$$

$$= x_n^2 - n\bar{x}_n^2 + \frac{1}{n-1}(n^2\bar{x}_n^2 - 2n\bar{x}_n x_n + x_n^2) \tag{15}$$

$$= \frac{1}{n-1}(nx_n^2 + n\bar{x}_n^2 - 2n\bar{x}_n x_n) \tag{16}$$

$$= \frac{n}{n-1}(\bar{x}_n - x_n)^2 \tag{17}$$

after which we have the recurrence formula for the sum of squares:

$$SS_n = SS_{n-1} + \frac{n}{n-1}(\bar{x}_n - x_n)^2 \tag{18}$$

Accordingly, the recurrence formula for the sample standard deviation (square root of variance) is:

$$s_1 := 0 \tag{19}$$

$$s_n = \sqrt{\frac{1}{n-1}\left((n-2)s_{n-1}^2 + \frac{n}{n-1}(\bar{x}_n - x_n)^2\right)} \tag{20}$$

In conclusion, a typical pseudocode for running calculation of $\bar{x}$ and s indicators is as follows:

Require: `read_value`(): returns a new observation of x at each call

1: $\bar{x} \leftarrow$ `read_value`() ▷ Initializations
2: $s \leftarrow 0$
3: $n \leftarrow 1$
4: **repeat** ▷ Main loop
5: $n \leftarrow n + 1$
6: $x \leftarrow$ `read_value`()
7: $\bar{x} \leftarrow \frac{1}{n}\left((n-1)\bar{x} + x\right)$ ▷ Update sample mean
8: $s \leftarrow \sqrt{\frac{1}{n-1}\left((n-2)s^2 + \frac{n}{n-1}(\bar{x} - x)^2\right)}$ ▷ Update sample std. dev.
9: Perform operations on $\bar{x}$ and s
10: **until** exit condition is true

R implementation

The following R code implements the above pseudocode, with some care to make it fun to use and *pipe-ready*:

```r
rstats <- function(prev, x=NULL, echo=FALSE) {
  result <- list(mean=NA, sd=0, n=0)
  if (!is.list(prev)) {
    v <- prev
    result$mean <- v
    result$n <- 1
  } else {
    n <- prev$n + 1
    m <- 1/n * ((n-1) * prev$mean + x)
    s <- ifelse(n > 1, sqrt(1/(n-1) * ((n-2) * prev$sd^2 + n/(n-1) * (m - x)^2)), 0)
    result$mean <- m
    result$sd <- s
    result$n <- n
  }
  if (echo)
    cat(sprintf("n: %d, mean: %.2f, sd: %.2f\n",
                result$n, result$mean, result$sd))
  invisible(result)
}
```

Simple use:

```r
rstats(10) %>% rstats(11) %>% rstats(12, echo=T)
```

```
n: 3, mean: 11.00, sd: 1.00
```

To be compared with:

```r
v <- 10:12
cat(sprintf("mean: %.2f, sd: %.2f, n: %d\n", mean(v), sd(v), length(v)))
```

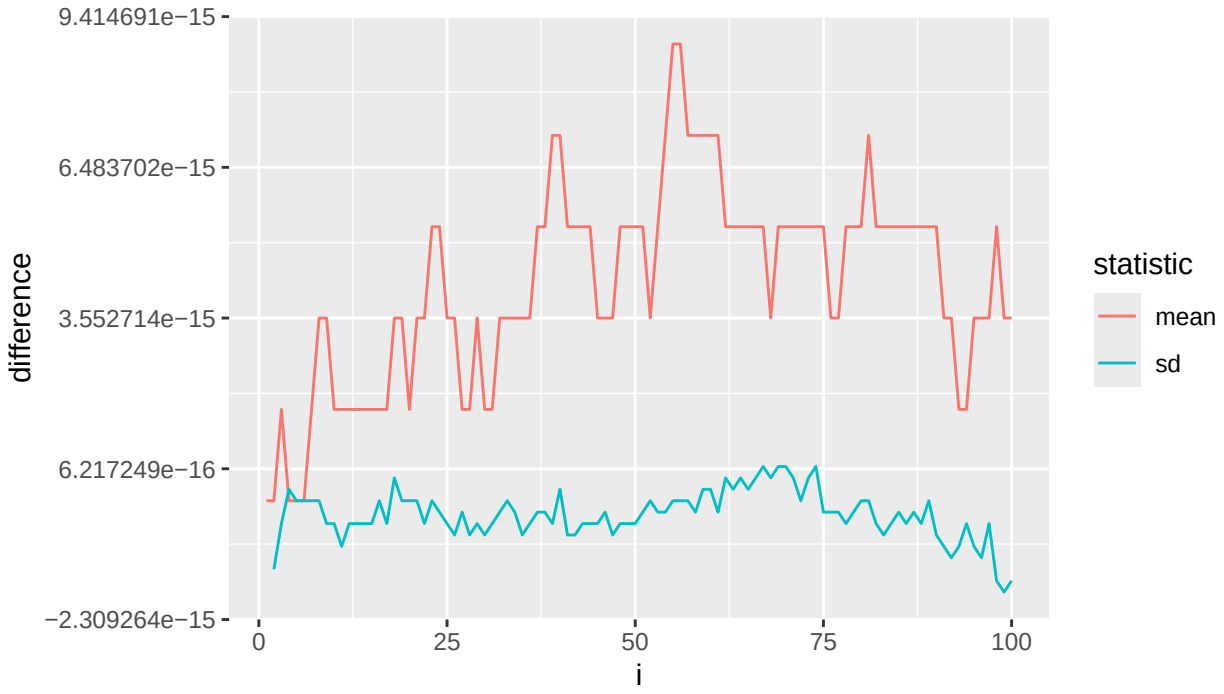
```
mean: 11.00, sd: 1.00, n: 3
```

Thanks to how we built the `rstats` function, we can use it in a `dplyr` pipeline using `purrr::accumulate()`:

```r
set.seed(0)
df <- tibble(
  i=1:100,
  x=rnorm(100, mean=10, sd=2)
```

```
) %>%
  mutate(
    # Warn: the first element of the list is the initial value of the
    # accumulator
    # This creates a list column with the stats at each step
    stats=accumulate(
      x,
      ~ rstats(.x, .y),
      .init=list(mean=first(x), sd=0, n=0))[2:(n()+1)],
    # usual cumulative stats:
    mean = cummean(x),
    sd = imap_dbl(x, ~ sd(x[1:.y]))
  ) %>%
  # unnest the list column
  hoist(stats, rmean=1, rsd=2) %>%
  mutate(
    mean = mean - rmean,
    sd = sd - rsd
  ) %>%
  pivot_longer(cols=c(mean, sd), names_to="statistic", values_to="difference")

df %>%
  ggplot(aes(x=i, y=difference)) +
  geom_line(aes(color=statistic))
```



So the recursion formulas *do suffer* from numerical precision, but the difference is not that big.

Cpp implementation

Much more interesting is the C/C++ implementation of the same algorithm. The following code is a simple, pure C implementation, plus a Rcpp wrapper to make it available in R with an interface similar to the R version:

```
#include "Rcpp.h"
using namespace Rcpp;


// Pure C version. n, mean and sd are IN/OUT parameters:
void rstats_c(double x, unsigned long int *n, double *mean, double *sd) {
  if (*n == 0) {
    *mean = x;
    *sd = 0;
    *n += 1;
  } else {
    *n += 1;
```

```
    double m = 1.0/ *n * ((*n-1) * (*mean) + x);
    double s = sqrt(1.0/(*n-1.0) * ((*n-2) * pow(*sd, 2) +
               *n/(*n-1.0) * pow(m - x, 2)));
    *mean = m;
    *sd = s;
  }
}

//[[Rcpp::export]]
List rstats(List &prev, double x) {
  double mean = prev["mean"];
  double sd = prev["sd"];
  unsigned long int n = prev["n"];
  rstats_c(x, &n, &mean, &sd);
  return List::create(
    Named("n") = n,
    Named("mean") = mean,
    Named("sd") = sd
  );
}
```

Now we can use the C++ version in R, paying attention that the C++ version interface is less smarter and it needs to have an initial zeroed list as input on the frst call:

```
rstats(list(mean = 0, sd=0, n=0), 10) %>% rstats(11) %>% rstats(12)
```

```
$n
[1] 3

$mean
[1] 11

$sd
[1] 1
```